

# Infrared finiteness and analyticity properties of the loop-loop scattering amplitudes in gauge theories

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We shall discuss about the infrared finiteness and some analyticity properties of the loop-loop scattering amplitudes in gauge theories, when going from Minkowskian to Euclidean theory, and we shall see how they can be related to the still unsolved problem of the  $s$ -dependence of the hadron-hadron total cross-sections.

Differently from the parton-parton scattering amplitudes, which are known to be affected by infrared (IR) divergences, the elastic scattering amplitude of two colourless states in gauge theories, e.g., two  $q\bar{q}$  meson states, is expected to be an IR-finite physical quantity. It was shown in Refs. [1,2,3] that the high-energy meson-meson elastic scattering amplitude can be approximately reconstructed by first evaluating, in the eikonal approximation, the elastic scattering amplitude of two  $q\bar{q}$  pairs (usually called “*dipoles*”), of given transverse sizes  $\vec{R}_{1\perp}$  and  $\vec{R}_{2\perp}$  respectively, and then averaging this amplitude over all possible values of  $\vec{R}_{1\perp}$  and  $\vec{R}_{2\perp}$  with two proper squared wave functions  $|\psi_1(\vec{R}_{1\perp})|^2$  and  $|\psi_2(\vec{R}_{2\perp})|^2$ , describing the two interacting mesons. The high-energy elastic scattering amplitude of two *dipoles* is governed by the correlation function of two Wilson loops  $\mathcal{W}_1$  and  $\mathcal{W}_2$ , which follow the classical straight lines for quark (antiquark) trajectories:

$$\mathcal{T}_{ll}(s, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv -i 2s \int d^2 \vec{z}_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} \left[ \frac{\langle \mathcal{W}_1 \mathcal{W}_2 \rangle}{\langle \mathcal{W}_1 \rangle \langle \mathcal{W}_2 \rangle} - 1 \right], \quad (1)$$

where  $s$  and  $t = -\vec{q}_\perp^2$  ( $\vec{q}_\perp$  being the tranferred momentum) are the usual Mandelstam variables. More explicitly the Wilson loops  $\mathcal{W}_1$  and  $\mathcal{W}_2$  are so defined:

$$\mathcal{W}_1^{(T)} \equiv \frac{1}{N_c} \text{Tr} \left\{ \mathcal{P} \exp \left[ -ig \oint_{\mathcal{C}_1} A_\mu(x) dx^\mu \right] \right\},$$

$$\mathcal{W}_2^{(T)} \equiv \frac{1}{N_c} \text{Tr} \left\{ \mathcal{P} \exp \left[ -ig \oint_{\mathcal{C}_2} A_\mu(x) dx^\mu \right] \right\}, \quad (2)$$

where  $\mathcal{P}$  denotes the “*path ordering*” along the given path  $\mathcal{C}$ ;  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are two rectangular paths which follow the classical straight lines for quark  $[X_{(+)}(\tau)$ , forward in proper time  $\tau$ ] and antiquark  $[X_{(-)}(\tau)$ , backward in  $\tau$ ] trajectories, i.e.,

$$\begin{aligned} \mathcal{C}_1 &\rightarrow X_{(\pm 1)}^\mu(\tau) = z^\mu + \frac{p_1^\mu}{m} \tau \pm \frac{R_1^\mu}{2}, \\ \mathcal{C}_2 &\rightarrow X_{(\pm 2)}^\mu(\tau) = \frac{p_2^\mu}{m} \tau \pm \frac{R_2^\mu}{2}, \end{aligned} \quad (3)$$

and are closed by straight-line paths at proper times  $\tau = \pm T$ , where  $T$  plays the role of an IR cutoff, which must be removed at the end ( $T \rightarrow \infty$ ). Here  $p_1$  and  $p_2$  are the four-momenta of the two quarks and of the two antiquarks with mass  $m$ , moving with speed  $\beta$  and  $-\beta$  along, for example, the  $x^1$ -direction:

$$\begin{aligned} p_1 &= m(\cosh \frac{\chi}{2}, \sinh \frac{\chi}{2}, 0, 0), \\ p_2 &= m(\cosh \frac{\chi}{2}, -\sinh \frac{\chi}{2}, 0, 0), \end{aligned} \quad (4)$$

where  $\chi = 2 \text{arctanh} \beta$  is the hyperbolic angle between the two trajectories (+1) and (+2). Moreover,  $R_1 = (0, 0, \vec{R}_{1\perp})$ ,  $R_2 = (0, 0, \vec{R}_{2\perp})$  and  $z = (0, 0, \vec{z}_\perp)$ , where  $\vec{z}_\perp = (z^2, z^3)$  is the impact-parameter distance between the two loops in the transverse plane.

It is convenient to consider also the correlation function of two Euclidean Wilson loops  $\tilde{\mathcal{W}}_1$  and  $\tilde{\mathcal{W}}_2$  running along two rectangular paths  $\tilde{\mathcal{C}}_1$  and  $\tilde{\mathcal{C}}_2$  which follow the following straight-line trajectories:

$$\tilde{\mathcal{C}}_1 \rightarrow X_{E\mu}^{(\pm 1)}(\tau) = z_{E\mu} + \frac{p_{1E\mu}}{m} \tau \pm \frac{R_{1E\mu}}{2},$$

$$\tilde{\mathcal{C}}_2 \rightarrow X_{E\mu}^{(\pm 2)}(\tau) = \frac{p_{2E\mu}}{m}\tau \pm \frac{R_{2E\mu}}{2}, \quad (5)$$

and are closed by straight-line paths at proper times  $\tau = \pm T$ . Here  $R_{1E} = (0, \vec{R}_{1\perp}, 0)$ ,  $R_{2E} = (0, \vec{R}_{2\perp}, 0)$  and  $z_E = (0, \vec{z}_\perp, 0)$ . Moreover, in the Euclidean theory we *choose* the four-vectors  $p_{1E}$  and  $p_{2E}$  to be:

$$\begin{aligned} p_{1E} &= m(\sin \frac{\theta}{2}, 0, 0, \cos \frac{\theta}{2}), \\ p_{2E} &= m(-\sin \frac{\theta}{2}, 0, 0, \cos \frac{\theta}{2}), \end{aligned} \quad (6)$$

where  $\theta \in [0, \pi]$  is the angle formed by the two trajectories (+1) and (+2) in Euclidean four-space.

Let us introduce the following notations for the normalized correlators  $\langle \mathcal{W}_1 \mathcal{W}_2 \rangle / \langle \mathcal{W}_1 \rangle \langle \mathcal{W}_2 \rangle$  in the Minkowskian and in the Euclidean theory, in the presence of a *finite* IR cutoff  $T$ :

$$\mathcal{G}_M(\chi; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \frac{\langle \mathcal{W}_1^{(T)} \mathcal{W}_2^{(T)} \rangle}{\langle \mathcal{W}_1^{(T)} \rangle \langle \mathcal{W}_2^{(T)} \rangle}, \quad (7)$$

$$\mathcal{G}_E(\theta; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \frac{\langle \tilde{\mathcal{W}}_1^{(T)} \tilde{\mathcal{W}}_2^{(T)} \rangle_E}{\langle \tilde{\mathcal{W}}_1^{(T)} \rangle_E \langle \tilde{\mathcal{W}}_2^{(T)} \rangle_E}.$$

As already stated in Ref. [4], the two quantities in Eq. (7) are expected to be connected by the same analytic continuation in the angular variables and in the IR cutoff which was already derived in the case of Wilson lines [4,5,6], i.e.:

$$\begin{aligned} \mathcal{G}_M(\chi; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) &= \\ \mathcal{G}_E(\theta \rightarrow -i\chi; T \rightarrow iT; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}). \end{aligned} \quad (8)$$

Indeed it can be proved [7], simply by adapting step by step the proof derived in Ref. [4] from the case of Wilson lines to the case of Wilson loops, that the analytic continuation (8) is an *exact* result, i.e., not restricted to some order in perturbation theory or to some other approximation, and is valid both for the Abelian and the non-Abelian case.

As we have said above, the loop-loop correlation functions (7), both in the Minkowskian and in the Euclidean theory, are expected to be IR-*finite* quantities, i.e., to have finite limits when  $T \rightarrow \infty$ , differently from what happens in the

case of Wilson lines. One can then define the following loop-loop correlation function in the Minkowskian theory with the IR cutoff removed,

$$\begin{aligned} \mathcal{C}_M(\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) &\equiv \\ \lim_{T \rightarrow \infty} \left[ \mathcal{G}_M(\chi; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) - 1 \right], \end{aligned} \quad (9)$$

and the corresponding quantity in the Euclidean theory,  $\mathcal{C}_E(\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})$ .

As a pedagogic example to illustrate these considerations, we shall consider the simple case of QED, in the so-called *quenched* approximation, where vacuum polarization effects, arising from the presence of loops of dynamical fermions, are neglected. In this approximation, the calculation of the normalized correlators (7) can be performed exactly (i.e., without further approximations) both in Minkowskian and in Euclidean theory and one finds that [7] i) the two quantities  $\mathcal{G}_M$  and  $\mathcal{G}_E$  are indeed connected by the analytic continuation (8), and ii) the two quantities are finite in the limit when the IR cutoff  $T$  goes to infinity:

$$\begin{aligned} \mathcal{C}_M(\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) &= \\ \exp \left[ -i4e^2 \coth \chi \, t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right] - 1, \end{aligned} \quad (10)$$

$$\begin{aligned} \mathcal{C}_E(\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) &= \\ \exp \left[ -4e^2 \cot \theta \, t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \right] - 1, \end{aligned} \quad (11)$$

where

$$\begin{aligned} t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) &\equiv \\ \frac{1}{8\pi} \ln \left( \frac{|\vec{z}_\perp + \frac{\vec{R}_{1\perp}}{2} + \frac{\vec{R}_{2\perp}}{2}| |\vec{z}_\perp - \frac{\vec{R}_{1\perp}}{2} - \frac{\vec{R}_{2\perp}}{2}|}{|\vec{z}_\perp + \frac{\vec{R}_{1\perp}}{2} - \frac{\vec{R}_{2\perp}}{2}| |\vec{z}_\perp - \frac{\vec{R}_{1\perp}}{2} + \frac{\vec{R}_{2\perp}}{2}|} \right) \end{aligned} \quad (12)$$

As shown in Ref. [7], the results (10) and (11) can be used to derive the corresponding results in the case of a non-Abelian gauge theory with  $N_c$  colours, up to the order  $\mathcal{O}(g^4)$  in perturbation theory (see also Refs. [8,9]):

$$\begin{aligned} \mathcal{C}_M(\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})|_{g^4} &= \\ -2g^4 \left( \frac{N_c^2 - 1}{N_c^2} \right) \coth^2 \chi \, [t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})]^2, \end{aligned} \quad (13)$$

$$\begin{aligned} \mathcal{C}_E(\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})|_{g^4} &= \\ 2g^4 \left( \frac{N_c^2 - 1}{N_c^2} \right) \cot^2 \theta \, [t(\vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp})]^2. \end{aligned} \quad (14)$$

We stress the fact that both the Minkowskian quantities (10) and (13) and the Euclidean quantities (11) and (14) are IR finite when  $T \rightarrow \infty$ , differently from the corresponding quantities constructed with Wilson lines, which were evaluated in Ref. [5] (see also Ref. [10]).

It is also important to notice that the two quantities (10) and (11), as well as the two quantities (13) and (14), obtained *after* the removal of the IR cutoff ( $T \rightarrow \infty$ ), are still connected by the usual analytic continuation in the angular variables only:

$$\mathcal{C}_M(\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = \mathcal{C}_E(\theta \rightarrow -i\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}). \quad (15)$$

This is a highly non-trivial result, whose general validity is discussed in Ref. [7]. (Indeed, the validity of the relation (15) has been also recently verified in Ref. [9] by an explicit calculation up to the order  $\mathcal{O}(g^6)$  in perturbation theory.)

As said in Ref. [7], if  $\mathcal{G}_M$  and  $\mathcal{G}_E$  are analytic functions of  $T$  in the whole complex plane and if  $T = \infty$  is an “eliminable singular point” [i.e., the finite limit (9) exists when letting the *complex* variable  $T \rightarrow \infty$ ], then, of course, the analytic continuation (15) immediately derives from Eq. (8), when letting  $T \rightarrow +\infty$ . (For example, if  $\mathcal{G}_M$  and  $\mathcal{G}_E$  are analytic functions of  $T$  and they are bounded at large  $|T|$ , then  $T = \infty$  is an “eliminable singular point” for both of them.) But the same result (15) can also be derived under weaker conditions. For example, let us assume that  $\mathcal{G}_E$  is a bounded analytic function of  $T$  in the sector  $0 \leq \arg T \leq \frac{\pi}{2}$ , with finite limits along the two straight lines on the border of the sector:  $\mathcal{G}_E \rightarrow G_{E1}$ , for  $(\text{Re}T \rightarrow +\infty, \text{Im}T = 0)$ , and  $\mathcal{G}_E \rightarrow G_{E2}$ , for  $(\text{Re}T = 0, \text{Im}T \rightarrow +\infty)$ . And, similarly, let us assume that  $\mathcal{G}_M$  is a bounded analytic function of  $T$  in the sector  $-\frac{\pi}{2} \leq \arg T \leq 0$ , with finite limits along the two straight lines on the border of the sector:  $\mathcal{G}_M \rightarrow G_{M1}$ , for  $(\text{Re}T \rightarrow +\infty, \text{Im}T = 0)$ , and  $\mathcal{G}_M \rightarrow G_{M2}$ , for  $(\text{Re}T = 0, \text{Im}T \rightarrow -\infty)$ . We can then apply the “Phragmén–Lindelöf theorem” to state that  $G_{E2} = G_{E1}$  and  $G_{M2} = G_{M1}$ . Therefore, also in this case, the analytic continuation (15) immediately derives from Eq. (8) when  $T \rightarrow \infty$ .

The relation (15) has been extensively used in the literature in order to address, from a non-perturbative point of view, the still unsolved problem of the asymptotic  $s$ -dependence of hadron-hadron elastic scattering amplitudes and total cross sections [8,11,12,13,14]. (It has been also recently proved in Ref. [9], by an explicit perturbative calculation, that the loop-loop scattering amplitude approaches, at sufficiently high energy, the BFKL-*pomeron* behaviour [15].)

An independent non-perturbative approach would be surely welcome and could be provided by a direct lattice calculation of the loop-loop Euclidean correlation functions. This would surely result in a considerable progress along this line of research.

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